

Comments on the boundary scattering phase

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ABSTRACT: We present a simple solution to the crossing equation for an open string world-sheet reflection matrix, with boundaries preserving a $SU(1|2)^2$ residual symmetry, which constrains the boundary dressing factor. In addition, we also propose an analogous crossing equation for the dressing factor where extra boundary degrees of freedom preserve a $SU(2|2)^2$ residual symmetry.

KEYWORDS: Integrable Field Theories, Extended Supersymmetry, AdS-CFT Correspondence, Gauge-gravity correspondence.

The exciting discovery of integrable structures in the planar $\mathcal{N} = 4$ super Yang-Mills [1–3]¹ has allowed for the determination of the perturbative spectrum for single trace operators. Particularly, in the limit of infinitely long operators [5–8], some exact all-loop results, which are also valid in the strong coupling limit, have been obtained. They allow precise comparisons with the energy spectrum of the non-interacting closed strings in $AdS_5 \times S^5$ carrying an infinitely large angular momentum, as prescribed by AdS/CFT correspondence [9]. In such infinite limit, physical content of the theory is the spectrum of asymptotic states and their scattering matrix, and as a hallmark of integrable system, it is highly constrained by the residual symmetry present. The relevant asymptotic Bethe equations then arise from imposing periodic boundary conditions. In fact, the residual symmetry proved to be sufficient to determine the two-particle scattering matrix between the elementary excitations up to an overall scalar factor [7]. Such overall scalar factor, known as “dressing factor” in the literature, plays an important role in interpolating the weak-strong coupling spectrum of the gauge/string correspondence [10–12]. Unconstrained by the residual symmetry, the dressing factor however needs to obey an extra symmetry known as “crossing” imposed on the aforementioned two-particle scattering matrix [11–13]. The crossing symmetry existing here also ensures the bound state of an elementary excitation and its anti-particle² scatters trivially [7].

It is also possible to consider certain operators in $\mathcal{N} = 4$ super Yang-Mills for which boundaries are present in the corresponding spin chains. This class of operators typically arises when one introduces extra D3-branes in the bulk where open strings can end on. As such open string becomes infinitely extended, locally near its end points, the system can be regarded as a semi-infinite configuration and the reflection of bulk excitations on the boundary has to be considered. Analogous to the bulk scattering matrix, the boundary reflection matrix can be determined up to an overall scalar factor by the residual symmetries preserving certain class of integrable boundaries [14].³ Moreover the overall scalar, sometimes referred as “boundary dressing factor”, can again be further restricted by analogous crossing symmetry conditions.

The main objective of this note is to study these crossing symmetry conditions in two different cases depending on the residual symmetries preserved. Here, like in the usual relativistic integrable theories [16], the boundary crossing symmetry condition relates the bulk and the boundary scattering matrices [14]. By specifying the boundary crossing transformation in relation to the bulk one, we shall point out a connection between bulk and boundary dressing factors and propose a simple solution to the boundary crossing equation derived in [14] for one of the two boundary conditions. Furthermore, we shall also present the crossing equation constraining the boundary dressing factor for the other boundary condition.

In general, the elementary excitation in the asymptotic spin-chain is known as “magnon”, and it can be specified by its definite momentum p , energy E and flavor.

¹See also [4] for more comprehensive reference list.

²Sometimes referred as the “singlet” states with respect to the residual symmetry algebra [7].

³See [15] for an earlier discussion on integrable boundary conditions for open strings in $AdS_5 \times S^5$.

The magnons of different flavor arise as they combine to form a short multiplet under the residual symmetry group the spin chain ground state preserves. For the infinite spin chain, as considered in [7], this is given by a centrally extended supergroup $PSU(2|2)^2 \times R^3 \cong SU(2|2)^2 \times R^2$ which preserves the ground state consisting exclusively of the complex adjoint scalar Z s. The energy of magnon E is then identified with one of the three central charges, and can be related to the magnon momentum by the dispersion relation:

$$E^2 - 16g^2 \sin^2 \frac{p}{2} = 1. \tag{1}$$

Here the coupling g is related to the 't Hooft coupling λ as $16\pi^2 g^2 = \lambda$.

The unusual relation (1) differs from the usual standard relativistic one or the one for lattice vibrations (e.g. phonons), but rather shares features of both. In fact, it describes a complex torus with two non-trivial circles [11, 13]. The first or “real” circle is given by shifting the magnon momentum $p \rightarrow p + 4\pi\mathbb{Z}$; whereas the second or “imaginary” circle arises when we regard $4g \sin \frac{p}{2}$ instead of p as the relevant relativistic momentum. Then for purely imaginary $4g \sin \frac{p}{2}$, the equation $E^2 + (i4g \sin \frac{p}{2})^2 = 1$ describes a unit circle. Therefore the magnon energy E and “momentum” $4g \sin \frac{p}{2}$ are in fact defined on a torus and can be uniformized and expressed in terms of Jacobi elliptic functions. We can also introduce an alternative set of complex variables known as the “spectral parameters” x^\pm , which are more convenient for our purposes and we shall mostly use subsequently. They are related to the momentum p and the energy E through:

$$e^{ip} = \frac{x^+}{x^-}, \quad E = \frac{1}{2} + \frac{ig}{x^+} - \frac{ig}{x^-}, \tag{2}$$

and they are subject to the constraint:

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}. \tag{3}$$

Combining (2) and (3) directly reproduces the dispersion relation (1). Again the x^\pm should be uniformized on the torus and we shall later give the explicit expressions in terms of Jacobi elliptic functions.

The asymptotic spin chains with boundaries turn up when studying the $\mathcal{N} = 4$ super Yang-Mills operators dual to D3-branes with open string excitations. In the $AdS_5 \times S^5$ background, such D3-branes [17] can be chosen to wrap a holomorphic surface within the S^5 .⁴ For instance, for holomorphic surfaces $Y = 0$ or $Z = 0$ the D3-branes are three-spheres of maximum size within S^5 and are usually called “maximal giant gravitons”. The perturbative computations in the dual field theory at weak coupling and the classical sigma model at strong coupling, have both shown that maximal giant gravitons provide integrable boundary conditions for the open string sigma model [18, 19, 15, 14]. Depending on the relative orientations of the angular momentum of the giant graviton and the open string ground state, two different cases can be considered. In one case, we can take the $Y = 0$

⁴The S^5 is then given in our conventions by the surface $|Z|^2 + |W|^2 + |Y|^2 = 1$.

giant graviton and let the open string ground state carry angular momentum along the Z direction. The operator corresponding to this configuration is [20]:

$$\mathcal{O}_Y = \epsilon_{j_1, j_2, \dots, j_{N-1}, j_N}^{i_1, i_2, \dots, i_{N-1}, i_N} Y_{i_1}^{j_1} Y_{i_2}^{j_2} \dots Y_{i_{N-1}}^{j_{N-1}} (ZZ \dots ZZ)_{i_N}^{j_N}, \quad (4)$$

where $i_n, j_n, n = 1, 2 \dots, N$ are the $SU(N)$ color indices. Essentially, one replaces the last entry in the operator $\det Y$ which is dual to the $Y = 0$ giant graviton by an infinite chain of Z s. As explained in details in [14], in order to preserve both Z s and Y s in the ground state (4), the residual symmetry group is reduced to $SU(1|2)^2 \subset PSU(2|2)^2 \ltimes R^3$. The elementary magnon which transforms as $(\mathbf{2}_B, \mathbf{2}_F)$ under each copy of $PSU(2|2) \ltimes R^3$ can now be expressed as an irreducible multiplet under each copy of $SU(1|2)$.

Alternatively, we can take a $Z = 0$ maximal giant graviton, with an open string ground state carrying angular momentum along the Z direction [14]

$$\mathcal{O}_Z(\chi_L, \chi_R) = \epsilon_{j_1, j_2, \dots, j_{N-1}, j_N}^{i_1, i_2, \dots, i_{N-1}, i_N} Z_{i_1}^{j_1} Z_{i_2}^{j_2} \dots Z_{i_{N-1}}^{j_{N-1}} (\chi_L Z Z \dots Z Z \chi_R)_{i_N}^{j_N}. \quad (5)$$

Boundary impurities are added to prevent the factorization into a determinant plus a single trace [21]. In this case, the full $PSU(2|2)^2 \ltimes R^3$ residual symmetry group preserves the string of Z s and the Z s in the determinant. The elementary magnons as well as the boundary degrees of freedom will transform in the fundamental representation (but carry different central charges) of the extended $SU(2|2)^2$.

Let us first recall the situation without boundary conditions. In such infinite asymptotic spin chain, the elementary magnons propagate freely apart from pairwise scattering, the physical content of the theory is therefore the two body scattering matrix $S(x_1, x_2)$. With the presence of an additional boundary, one also needs to take into account the scattering between the magnon and the boundary and encode such interaction in a “reflection matrix $\mathcal{R}(x)$ ”.⁶ It was shown in [7] by demanding the invariance of the scattering matrix $S(x_1, x_2) \equiv S(x_1^\pm, x_2^\pm)$ under the residual symmetry algebra $\mathfrak{psu}(2|2)^2 \ltimes \mathbb{R}^3$, that it can be constrained up to an overall scalar. This is a hallmark of an integrable system and $S(x_1, x_2)$ takes the following schematic form:

$$S_{\text{full}}(x_1, x_2) = S_0^2(x_1, x_2) \left(\hat{S}_{\mathfrak{su}(2|2)}(x_1, x_2) \otimes \hat{S}_{\mathfrak{su}'(2|2)}(x_1, x_2) \right). \quad (6)$$

Here $\hat{S}_{\mathfrak{su}(2|2)}(x_1, x_2)$ and $\hat{S}_{\mathfrak{su}'(2|2)}(x_1, x_2)$ are flavor dependent parts and are uniquely fixed by each copy of $\mathfrak{su}(2|2) \ltimes \mathbb{R}^2$ respectively, and non-trivially satisfy the Yang-Baxter and unitarity equations. This further confirms the integrable structure of the theory. Whereas the remaining overall scalar factor $S_0(x_1, x_2)^2$ is given by⁷

$$S_0(x_1, x_2)^2 = \frac{(x_1^+ - x_2^-) \left(1 - \frac{1}{x_1^- x_2^+}\right)}{(x_1^- - x_2^+) \left(1 - \frac{1}{x_1^+ x_2^-}\right)} \frac{1}{\sigma^2(x_1, x_2)}, \quad (7)$$

⁵The three central charges are shared between both copies of $PSU(2|2) \ltimes R^3$.

⁶Sometimes in the literature one refers $S(x_1, x_2)$ as the “bulk” scattering matrix to distinguish from the “boundary” scattering matrix $\mathcal{R}(x)$ [16].

⁷Here we follow the conventions as in [11] and [14].

the function $\sigma(x_1, x_2) \equiv \sigma(x_1^\pm, x_2^\pm)$ is usually referred to in the literature as the “dressing factor” [6, 10–12]. To determine $\sigma(x_1, x_2)$ additional dynamical constraints, such as crossing symmetry in a relativistic theory [22] which interchanges particle/anti-particle (and perhaps higher loop computations), are required. Despite the non-standard dispersion relation for the elementary magnon (1), it was demonstrated in [13] that crossing symmetry can also be implemented provided the dressing factor satisfies the crossing equations:

$$\sigma(\bar{x}_1, x_2)\sigma(x_1, x_2) = \frac{x_2^-}{x_2^+} f(x_1, x_2), \tag{8}$$

$$\sigma(x_1, \bar{x}_2)\sigma(x_1, x_2) = \frac{x_1^+}{x_1^-} f(x_1, x_2), \tag{9}$$

where $\bar{x}_{1,2}$ are the “crossed” spectral parameters which we shall explain momentarily, and the function $f(x_1, x_2)$ is given by

$$f(x_1, x_2) \equiv \frac{(x_1^- - x_2^+)(1 - 1/x_1^+ x_2^+)}{(x_1^- - x_2^-)(1 - 1/x_1^+ x_2^-)} = \frac{(x_1^- - x_2^+)(1 - 1/x_1^- x_2^-)}{(x_1^+ - x_2^+)(1 - 1/x_1^+ x_2^-)}. \tag{10}$$

These equations also need to be supplemented with the unitarity condition

$$\sigma(x_1, x_2)\sigma(x_2, x_1) = 1. \tag{11}$$

Let us now explain how crossing symmetry can be implemented in the context of magnon scattering following [11, 13]. As discussed earlier, the magnon energy and momentum or equivalently the spectral parameters x^\pm are in fact defined on a torus. Therefore we can parametrize them, subject to the dispersion relation (1) or (3) in terms of Jacobi elliptic functions [11]:

$$p(z) = 2\text{am}(z, k), \quad \sin \frac{p(z)}{2} = \text{sn}(z, k), \quad E(z) = \text{dn}(z, k). \tag{12}$$

Here the elliptic modulus $k = 4ig$ can be taken fixed in a conformal field theory, so that p and E can be regarded as functions of the complex parameter z called “generalized rapidity”.⁸ Simple manipulation of elliptic functions identities shows that the spectral parameters are given by,

$$x^\pm(z) = \frac{1}{4g} \left(\frac{\text{cn}(z, k)}{\text{sn}(z, k)} \pm i \right) (1 + \text{dn}(z, k)). \tag{13}$$

The two circles of the complex torus can be quantified by the shifts $z \rightarrow z \pm 2\omega_1$ and $z \rightarrow z \pm 2\omega_2$ with

$$\omega_1 = 2K(k^2), \quad \omega_2 = 2iK(1 - k^2) - 2K(k^2), \tag{14}$$

where $K(k^2)$ is the complete elliptic integral of first kind.⁹ For elliptic modulus $k^2 = -16g^2 \in \mathbb{R}$, $\text{Im}(\omega_1) = \text{Re}(\omega_2) = 0$ and ω_1 and ω_2 can be identified with the half-period

⁸The terminology here is drawn from the relativistic case, where the parametrization $\epsilon = m \cosh \theta, p = m \sinh \theta, \theta \in \mathbb{R}$ satisfies the dispersion relation $\epsilon^2 - p^2 = m^2$.

⁹Our convention is $K(k^2) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$.

of real and imaginary circles respectively. The complex torus can then be defined on the generalized rapidity z -plane by the domain $\{|\operatorname{Re}(z)| \leq \omega_1, |\operatorname{Im}(z)| \leq \omega_2\}$. However, one should stress that $(E(z), p(z))$ are not both real in the entire domain and the function $S(z_1, z_2)$ should be regarded as an analytic continuation of the S-matrix with real E and p (real values of the generalized rapidity) [23, 24].

Now let us consider how $x^\pm(z)$ transform under the translations on the z -plane. One can first verify that, along the real axis, the shift $z \pm \omega_1$ leaves $x^\pm(z)$ invariant. However along the imaginary axis the shift $z \pm \omega_2$ transforms $x^\pm(z)$ as

$$\text{Crossing : } x^\pm(z) \rightarrow x^\pm(z \pm \omega_2) = \bar{x}^\pm(z) = \frac{1}{x^\pm(z)}. \tag{15}$$

Such transformation (15) is the ‘‘crossing transformation’’ defined in terms of the spectral parameters given in [13]. Here the notation $\bar{x}^\pm(z) \equiv \bar{x}^\pm$ are the crossed spectral parameters introduced earlier in (8) and (9). In terms of the magnon energy $E(z)$ and momentum $p(z)$, (15) corresponds to the transformation $(E(z), p(z)) \rightarrow (-E(z), -p(z))$ which is precisely how crossing transformation acts in relativistic theory, hence the terminology. One should note that in terms of the spectral parameters x^\pm , the ‘‘double crossing’’, i.e. applying (15) twice, appears to be a trivial map. However, on z -plane this is a non-trivial translation $z \rightarrow z + 2\omega_2$ going round the imaginary circle of the complex torus once. A non-trivial monodromy can be seen from the fact that the ratio $\sigma(\bar{x}_1, x_2)/\sigma(x_1, x_2) = f(\bar{x}_1, x_2)/f(x_1, x_2) \neq 1$. Also because of that, it turns out that the sign (orientation) in the shift $z \rightarrow z \pm \omega_2$ implementing the crossing transformation matters. The consistency of crossing transformations (8)–(9) with the unitarity condition demands us to define the shifts in the two arguments of the bulk S-matrix with ‘‘opposite’’ signs.¹⁰ That is, for the two arguments in $\sigma(x_1, x_2) \equiv \sigma(x(z_1), x(z_2))$, there are two equivalent ways in which crossing transformation can act:

$$(+, -) : z_1 \rightarrow z_1 + \omega_2, \quad z_2 \rightarrow z_2 - \omega_2, \tag{16}$$

$$(-, +) : z_1 \rightarrow z_1 - \omega_2, \quad z_2 \rightarrow z_2 + \omega_2. \tag{17}$$

For definiteness, we will adhere to the convention (16) for the rest of this note. A class of consistent crossing symmetric dressing factors satisfying (8), (9) and (11) were found in [11]. Moreover a unique special solution which correctly reproduces the weak coupling gauge theory results was further singled out in [12].

Finally, one notes the discrete parity transformation can be imposed on the z -plane as $z \rightarrow -z$, such that:

$$\text{Parity : } x^\pm(-z) = -x^\mp(z). \tag{18}$$

In terms of the magnon energy and momentum, this yields $(E(-z), p(-z)) = (E(z), -p(z))$ as expected.

¹⁰A more correct way to put it is that to be consistent with the underlying Hopf-algebra, it is necessary to act on the first argument with the anti-pode S and on the second with the inverse S^{-1} [13], or vice versa.

The construction of the boundary reflection matrix $\mathcal{R}(x)$ proceeds in an almost identical way as for the bulk scattering matrix $S(x_1, x_2)$ [14]. For the $Y = 0$ case, its form can again be constrained up to an overall scalar factor by demanding its invariance under the residual $\mathfrak{su}(1|2)^2$ symmetry algebra. Schematically, the result is:

$$\mathcal{R}_R^{\text{full}}(x) = \mathcal{R}_{0R}^2(x) \hat{\mathcal{R}}_R^{\text{su}(1|2)}(x) \otimes \hat{\mathcal{R}}_R^{\text{su}'(1|2)}(x). \quad (19)$$

Here the subscript R denotes the scattering of a magnon with the right boundary.¹¹ $\hat{\mathcal{R}}_R^{\text{su}(1|2)}(x)$ and $\hat{\mathcal{R}}_R^{\text{su}'(1|2)}(x)$ denotes the flavor dependent parts, which are uniquely fixed to be

$$\hat{\mathcal{R}}_R^{\text{su}(1|2)}(x) = \begin{pmatrix} -\frac{x^-}{x^+} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (20)$$

whereas $\mathcal{R}_{0R}^2(x)$ is the boundary equivalent of the dressing factor.

To derive the relevant crossing symmetry equation for the boundary dressing factor $\mathcal{R}_{0R}(x)$, one can recall an alternative derivation for (8) and (9), which demands the trivial scattering between the $\mathfrak{psu}(2|2) \times \mathbb{R}^3$ singlet state, given explicitly by

$$|\mathbf{1}_{p, \bar{p}}\rangle = \frac{\alpha}{\gamma_p \gamma_{\bar{p}} \xi_p} \left(\frac{x^+}{x^-} - 1 \right) \epsilon_{ab} |\mathcal{Y}^- \mathcal{Y}^- \mathcal{Z}^+ \phi_p^a \phi_{\bar{p}}^b\rangle + \epsilon_{\alpha\beta} |\psi_p^\alpha \psi_{\bar{p}}^\beta\rangle, \quad (21)$$

and an elementary magnon in the bulk [7]. Similarly, if we demand that the singlet (21) scatters trivially with the right boundary (as depicted in the lower right corner of each picture in figure 1), the relevant scattering process can be described as

$$|\mathbf{1}_{p, \bar{p}}\rangle \rightarrow \mathcal{R}_R(p) S(p, -\bar{p}) \mathcal{R}_R(\bar{p}) |\mathbf{1}_{p, \bar{p}}\rangle = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^+}} S_0(p, -\bar{p}) \mathcal{R}_{0R}(x) \mathcal{R}_{0R}(\bar{x}) |\mathbf{1}_{-\bar{p}, -p}\rangle. \quad (22)$$

Symmetry under parity transformation of both, bulk and boundary scattering matrix, implies that the reflection in the left boundary will contribute with the same factor and revert the singlet to its original orientation. This leads to a boundary crossing-symmetry condition for the factor $\mathcal{R}_{0R}(p)$ [14]

$$\mathcal{R}_{0R}^2(p) \mathcal{R}_{0R}^2(\bar{p}) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}} \sigma^2(p, -\bar{p}). \quad (23)$$

This equation again needs to obey the unitarity constraint:

$$\mathcal{R}_{0R}^2(p) \mathcal{R}_{0R}^2(-p) = 1. \quad (24)$$

As we shall show momentarily, it will be crucial to consider the generalized rapidity z -plane coordinates and apply the crossing (15) transformation consistently in order to find the solution(s) to (23) and (24).

¹¹The reflection matrix $\mathcal{R}_L(x)$ for left boundary can be deduced from (19) using the parity symmetry of the problem, i.e. $\mathcal{R}_L(x(z)) = \mathcal{R}_R(x(-z))$ [14].

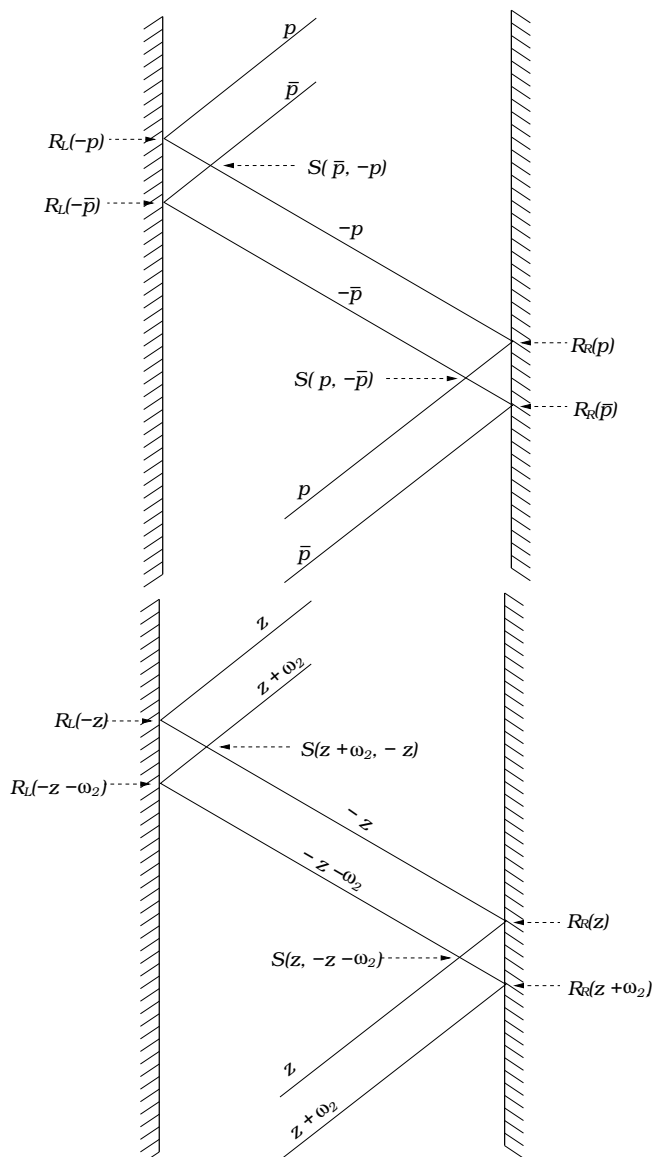


Figure 1: Scattering of a singlet state, in terms of momenta (left) and rapidities (right).

Essentially, one needs to specify how to implement the crossing transformations in eq. (23), when writing it in terms of the generalized rapidity z . The consistent choice of relative signs in the shifts would be the one such that the action of the reflection is just flipping the sign of the rapidity. This means that, as shown in figure 1 for instance, the argument of the boundary dressing factor for anti-particle \bar{p} , has to be opposite in sign to the second argument of the bulk dressing factor. Thus, if taking \bar{p} as $p(z + \omega_2)$ in the l.h.s.

of (23), the consistent choice is¹²

$$\mathcal{R}_{0R}^2(z)\mathcal{R}_{0R}^2(z + \omega_2) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(z, -z - \omega_2). \quad (25)$$

Before we present our proposed solution to the boundary crossing equation (25), let us first demonstrate that a different choice of relative shift signs would have been inconsistent with unitarity. Consider for example

$$\mathcal{R}_{0R}^2(z)\mathcal{R}_{0R}^2(z - \omega_2) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(z, -z - \omega_2). \quad (26)$$

Without loss of generality we can write the boundary dressing factor $\mathcal{R}_{0R}^2(z)$ as

$$\mathcal{R}_{0R}^2(z) = F(z)\sigma(-z, z), \quad (27)$$

where $\sigma(z_1, z_2)$ satisfies (8), (9) and (11). There are a few ways to motivate this. First in the so-called ‘‘Giant Magnon regime’’ [8, 23], the strong coupling computation using classical string theory [14] shows that

$$\mathcal{R}_{0R}^2(z) = e^{ig\theta_0(-z, z) + \mathcal{O}(1)}, \quad (28)$$

where $\log \sigma(z_1, z_2) = ig\theta_0(z_1, z_2) + \mathcal{O}(1)$ is the leading dressing phase calculated originally in [10, 8]. The ansatz (27) correctly captures the strong coupling result (28) provided $\log F(z) \rightarrow \mathcal{O}(1)$ as $g \rightarrow \infty$. Second, the factor $\sigma(p, -\bar{p})$ on the r.h.s. of (23) can be easily re-written in terms of $\sigma(-z, z)$ provided consistent crossing convention is taken, then we are left with conditions for $F(z)$.

With the ansatz (27), the boundary crossing equation (26) would be written as

$$F(z)F(z - \omega_2)\sigma(-z, z)\sigma(-z + \omega_2, z - \omega_2) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(z, -z - \omega_2). \quad (29)$$

The successive application of (8)–(9), allows one to write

$$\frac{\sigma(-z, z)\sigma(-z + \omega_2, z - \omega_2)}{\sigma^2(z, -z - \omega_2)} = \frac{f(-z + \omega_2, z)}{f(z, -z)^2 f(-z, z)} = \frac{(x^+ + \frac{1}{x^+})^3 (x^- + \frac{1}{x^-})(x^+ + x^-)^4}{16(1 + x^+ x^-)^4}. \quad (30)$$

To satisfy (29) we would need to impose

$$F(z)F(z - \omega_2) = \frac{16(1 + x^+ x^-)^4}{(x^+ + \frac{1}{x^+})^2 (x^- + \frac{1}{x^-})^2 (x^+ + x^-)^4}. \quad (31)$$

It is important to notice that the r.h.s. of (31) is crossing invariant. As a consequence,

$$F(z - 2\omega_2) = F(z), \quad (32)$$

¹²Here we have used the simplified notation $\sigma(z_1, z_2) \equiv \sigma(x(z_1), x(z_2))$, $\mathcal{R}_{0R}(z) \equiv \mathcal{R}_{0R}(x(z))$ and x^\pm as $x^\pm(z)$ to avoid overlong expressions.

i.e. the non-trivial factor due to double crossing the argument of $\mathcal{R}_{0R}(z)$ is completely accounted by the shift on the factor $\sigma(-z, z)$. Now if we apply the parity transformation $z \rightarrow -z$ to (31) and use unitarity condition $F(z)F(-z) = 1$, we get

$$\frac{F(-z - \omega_2)}{F(z)} = \frac{16(1 + x^+x^-)^4}{(x^+ + \frac{1}{x^+})^2(x^- + \frac{1}{x^-})^2(x^+ + x^-)^4}, \quad (33)$$

whereas imposing unitarity only on the original (31) gives

$$\frac{F(z)}{F(-z + \omega_2)} = \frac{16(1 + x^+x^-)^4}{(x^+ + \frac{1}{x^+})^2(x^- + \frac{1}{x^-})^2(x^+ + x^-)^4}. \quad (34)$$

However (32) would then tell us

$$\frac{16(1 + x^+x^-)^4}{(x^+ + \frac{1}{x^+})^2(x^- + \frac{1}{x^-})^2(x^+ + x^-)^4} = 1, \quad (35)$$

which is clearly not the case for arbitrary x^\pm ! This lead us conclude that equations (33) and (34) are in obvious contradiction. This means that with the choice of shift signs (26), boundary crossing symmetry and unitarity would be inconsistent with each other.

Now let us see that when adopting the other choice of shift signs (25), which we argued to be the consistent one, both boundary crossing equation and unitarity condition can be simultaneously solved with ease. Using once again the ansatz (27) we obtain

$$F(z)F(z + \omega_2)\sigma(-z, z)\sigma(-z - \omega_2, z + \omega_2) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}\sigma^2(z, -z - \omega_2). \quad (36)$$

According to (8) and (9),

$$\frac{\sigma(-z, z)\sigma(-z - \omega_2, z + \omega_2)}{\sigma(z, -z - \omega_2)^2} = \frac{1}{f(z, -z - \omega_2)f(z + \omega_2, -z - \omega_2)} = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}}. \quad (37)$$

Thus we obtain the following crossing symmetry and unitarity conditions imposed on $F(z)$

$$\begin{aligned} F(z)F(z + \omega_2) &= 1, \\ F(z)F(-z) &= 1, \end{aligned} \quad (38)$$

The system can be solved generally by $F(z) = \pm \exp(if_{\text{odd}}(p))$ where $f_{\text{odd}}(p)$ is an arbitrary odd function of the magnon momentum p , moreover at strong coupling when $g \rightarrow \infty$, $\log F(z) \rightarrow \mathcal{O}(1)$. By further comparing with the weak coupling expression in (4.60) of [14] for $\mathcal{R}_{0R}^2(z)$, this further requires that $F(z) \rightarrow -\exp(2ip)$ when $g \rightarrow 0$. Thus, the simplest solution for $\mathcal{R}_{0R}^2(z)$ satisfying the $Y = 0$ case crossing equation (23) and in agreement with the known strong/weak coupling results is:

$$\mathcal{R}_{0R}^2(z) = -\exp(2ip)\sigma(-z, z) \equiv -\exp(2ip(z))\sigma(x(-z), x(z)). \quad (39)$$

Let us briefly comment on the uniqueness of (39). Recall that the factor $\sigma(p, -\bar{p})$ entering in the r.h.s. of (23) is unambiguously interpreted as the unique bulk dressing factor identified

in [12]. We then showed that $\mathcal{R}_{0R}^2(z)$ is given in terms of $\sigma(-z, z)$ satisfying (8)–(9). Therefore, for consistency, we should also interpret $\sigma(-z, z)$ to have exactly the same functional form as the unique bulk dressing factor identified in [12]. A good way to verify our solution would be to explicitly calculate the leading semi-classical $\frac{1}{g}$ correction in $\mathcal{R}_{0R}^2(z)$ by generalizing the worldsheet approach in [28] to the situation with open boundary conditions.

We end this note by considering also the crossing equation for the case $Z = 0$. Here the form of the boundary reflection matrix $\mathcal{R}(x)$ is again fixed up to an overall scalar factor $\mathcal{R}_{0R}(x)$ by demanding its invariance under the residual $\mathfrak{su}(2|2)^2$ symmetry algebra [14],

$$\mathcal{R}_R^{\text{full}}(x) = \mathcal{R}_{0R}^2(x) \hat{\mathcal{R}}_R^{\text{su}(2|2)}(x) \otimes \hat{\mathcal{R}}_R^{\text{su}'(2|2)}(x). \quad (40)$$

Because of the extra boundary degree of freedom, $\hat{\mathcal{R}}_R^{\text{su}(2|2)}(x)$ and $\hat{\mathcal{R}}_R^{\text{su}'(2|2)}(x)$ are now 16×16 matrices. They are non-diagonal and their complexity is similar to that of $\hat{S}_{\mathfrak{su}(2|2)}(x_1, x_2)$. The explicit expressions for their components are presented in (3.42)–(3.46) of [14] and they have also been checked to satisfy the boundary Yang-Baxter equations.

As done in [14] for the case $Y = 0$, we can use the reflection of the singlet state (21) to derive a relevant crossing symmetry condition. Now from the factorizability, the scattering processes displayed in the right lower corner of figure 1 consist of the action of three 16×16 dimensional scattering matrices. After a computationally intense calculation we obtain, regardless of the flavor of the boundary impurity, the following result for reflection of the singlet state:

$$\mathcal{R}_R(p) S(p, -\bar{p}) \mathcal{R}_R(\bar{p}) | \mathbf{1}_{p, \bar{p}} \chi_R \rangle = \frac{x^- + \frac{1}{x^-}}{x^+ + \frac{1}{x^+}} h_B(p) S_0(p, -\bar{p}) \mathcal{R}_{0R}(x) \mathcal{R}_{0R}(\bar{x}) | \mathbf{1}_{-\bar{p}, -p} \chi_R \rangle, \quad (41)$$

where

$$h_B(p) = \frac{x^+(x_B - x^-)}{x^-(x_B - x^+)} \frac{1 + (x_B x^- x^+)^2}{(1 - (x_B x^+)^2)(1 - x^- x^+)}, \quad (42)$$

and $x_B = \frac{i}{2g}(1 + \sqrt{1 + 4g^2})$. This leads to a crossing-symmetry condition for the boundary dressing factor $\mathcal{R}_{0R}(p)$

$$\mathcal{R}_{0R}^2(p) \mathcal{R}_{0R}^2(\bar{p}) = \frac{x^+ + \frac{1}{x^+}}{x^- + \frac{1}{x^-}} \frac{\sigma^2(p, -\bar{p})}{h_B(p)^2}. \quad (43)$$

As a simple check for (43), one notices that in the leading $g \rightarrow \infty$ expansion (or more precisely, the giant magnon limit $x^\pm \sim e^{\pm ip/2}$), $h_B(x) \rightarrow \mathcal{O}(1)$ and then the classical dressing factor

$$\mathcal{R}_{0R}^2(z) = e^{ig\theta_0(-z, z) + 2ig\theta_0(\omega_1/2, z) + \mathcal{O}(1)}, \quad (44)$$

deduced in [14] using the “method of images” satisfies the classical limit of the crossing equation (43). Up to the factor $1/h_B(p)^2$, one recognizes the crossing symmetry equation of the $Y = 0$ case, which we showed can be simply solved by $\sigma(-z, z)$. Under crossing and unitarity conditions, the naive ansatz $\mathcal{R}_{0R}^2(z) = F(z)\sigma(-z, z)$ yields

$$\begin{aligned} F(z)F(z + \omega_2) &= \frac{1}{h_B(p(z))^2}, \\ F(z)F(-z) &= 1. \end{aligned} \quad (45)$$

and we have not attempted to solve this set of equations. One should also notice that $h_B(p(z))^2$ is not crossing invariant, therefore the function $F(z)$ still has a non-trivial monodromy, i.e. $F(z + 2\omega_2) \neq F(z)$, and then a non trivial crossing condition remains to be solved.

As a first step however, it would be very desirable to obtain the leading semiclassical correction to the boundary dressing factor in this $Z = 0$ case as in its closed string counterpart [25] and test if crossing symmetry condition (43) is obeyed (c.f. [26]). In particular, in contrast with the case $Y = 0$, here the boundary analogue of the magnon bound states exists [14, 27]. Therefore, it would be interesting to generalize the constructions of multi-soliton solutions in [28, 29], also their classical [30] and semiclassical scatterings [28] to the case with boundaries. Here the fluctuations around the soliton background would need to be subject to appropriate boundary conditions, this direction is currently under investigation.

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